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# Optimality conditions for a cone-dc vector optimization problem

新潟大学大学院自然科学研究科　山田修司 (Syuuji YAMADA)
Graduate School of Science and Technology, Niigata University
新潟大学大学院自然科学研究科　田中環 (Tamaki TANAKA)
Graduate School of Science and Technology, Niigata University
大阪大学大学院工学研究科　谷野哲三 (Tetsuzo TANINO)
Graduate School of Engineering, Osaka University

**Abstract.** In this paper, we propose necessary and sufficient optimality conditions for three types of solutions of a vector-valued cone-dc vector optimization problem in the case where the ordering cone is obtained as a convex polyhedral cone.

**Keywords:** Cone-dc function, Vector optimization problem, Optimality condition.

## 1 Introduction

An optimization problem including dc functions (difference of two convex functions) is called dc programming. Dc programming is one of the important subjects in global optimization and has been studied by Rosen [7], Avriel and Williams [1], Meyer [6], Ueing [9], Hillestad and Jacobsen [2], and Tuy [8]. It is known that many global optimization problems can be transformed into or approximated by dc programming problems. For dc programming problems, a necessary and sufficient optimality condition has been proposed by Hiriart [3].

The concept of vector-valued cone-dc function has been proposed by Hojo, Tanaka and Yamada [4]. Moreover, several properties of cone-dc function have been analysed by Yamada, Tanaka and Tanino [10]. In particular, it has been shown that every locally vector-valued cone-dc function having a compact convex domain can be rewritten as a vector-valued cone-dc function with the same domain. From such results, we notice that many vector optimization problems can be transformed into or approximated by vector-valued cone-dc vector optimization problems.

In this paper, we propose necessary and sufficient optimality conditions for three types of solutions of an unconstrained vector-valued cone-dc vector optimization problem by applying Hiriart's optimality condition in the case where the ordering cone is obtained as a convex polyhedral cone. By utilizing a penalty function method, the proposed optimality conditions adapt to a constrained vector-valued cone-dc vector optimization problem.

The organization of this paper is as follows: In Section 2, we introduce some notation and mathematical preliminaries in convex analysis. In Section 3, we propose optimality conditions for a cone-dc vector optimization problem.

## 2 Mathematical Preliminaries

Throughout this paper, we use the following notation: $\mathbb{R}$ denotes the set of all real numbers. For a natural number $m$, $\mathbb{R}^m$ denotes an $m$-dimensional Euclidean space. The origin of $\mathbb{R}^m$ is denoted by $\mathbf{0}_m$. Let $\mathbb{R}^m_+ := \{\boldsymbol{x} \in \mathbb{R}^m : \boldsymbol{x} \geq \mathbf{0}_m\}$. Given a vector $\boldsymbol{x} \in \mathbb{R}^m$, $\boldsymbol{x}^\top$ denotes the transposed vector of $\boldsymbol{x}$. For a subset $X \subset \mathbb{R}^m$, $\operatorname{int} X$ denotes the interior of $X$. Given a nonempty cone $D \subset \mathbb{R}^m$, $D^+$ denotes the positive polar cone of $D$, that is, $D^+ := \{\boldsymbol{u} \in \mathbb{R}^m : \boldsymbol{u}^\top \boldsymbol{x} \geq 0, \text{ for all } \boldsymbol{x} \in D\}$. Given a real-valued function $\alpha : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ and a positive real number $\varepsilon$, $\partial_\varepsilon \alpha(\boldsymbol{y})$ denotes the $\varepsilon$-subdifferential of $\alpha$ at $\boldsymbol{y}$, that is,

$$\partial_\varepsilon \alpha(\boldsymbol{y}) := \{\boldsymbol{a} \in \mathbb{R}^m : \boldsymbol{a}^\top(\boldsymbol{x} - \boldsymbol{y}) + \alpha(\boldsymbol{y}) - \varepsilon \leq \alpha(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^m\}.$$

Moreover, we use the following concepts of convex analysis.

**Definition 2.1** (**Real-valued dc function**) Let $X$ be a nonempty convex subset on $\mathbb{R}^m$. A function $\alpha : X \to \mathbb{R}$ is said to be *dc* on $X$ if there exist two real-valued convex functions $\beta, \gamma : X \to \mathbb{R}$ such that

$$\alpha(\boldsymbol{x}) = \beta(\boldsymbol{x}) - \gamma(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in X. \tag{1}$$

The representation (1) is called a *dc decomposition* of $\alpha$ on $X$. When $X = \mathbb{R}^m$, $F$ is simply called a *dc* function and the representation (1) is simply called a *dc decomposition.*

**Proposition 2.2** (See Hiriart [3]) *Let $\beta : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function and $\gamma : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ a convex proper l.s.c function. A point $\boldsymbol{x} \in$ (dom $\beta$) $\cap$ (dom $\gamma$) is a global minimizer of $\beta(\boldsymbol{x}) - \gamma(\boldsymbol{x})$ of $\mathbb{R}^m$ if and only if*

$$\partial_\varepsilon \gamma(\boldsymbol{x}) \subset \partial_\varepsilon \beta(\boldsymbol{x}) \quad \text{for each } \varepsilon > 0.$$

Let $C \subset \mathbb{R}^n$ be a convex polyhedral cone defined as

$$C := \left\{\boldsymbol{x} \in \mathbb{R}^n : \left(\boldsymbol{c}^i\right)^\top \boldsymbol{x} \geq 0,\ i = 1, \ldots, l\right\}.$$

We assume that $\operatorname{int} C \neq \emptyset$ and that $\dim\{\boldsymbol{c}^1, \ldots, \boldsymbol{c}^l\} = n$. Then, we note that $C$ is pointed.

Now, we define the order $\leq_C$ as follows:

$$\boldsymbol{x} \leq_C \boldsymbol{y} \text{ if } \boldsymbol{y} - \boldsymbol{x} \in C \text{ for } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

Moreover, we review the following concept for vector-valued functions.

**Definition 2.3** ($C$-**convexity**, see, e.g., Luc [5], Definition 6.1) Let $X \subset \mathbb{R}^m$ be nonempty and convex. A vector-valued function $g : X \to \mathbb{R}^n$ is said to be *$C$-convex* on $X$ if for each $\boldsymbol{x}^1, \boldsymbol{x}^2 \in X$ and $0 \leq \lambda \leq 1$, $g$ satisfies that

$$g((1-\lambda)\boldsymbol{x}^1 + \lambda \boldsymbol{x}^2) \leq_C (1-\lambda) g(\boldsymbol{x}^1) + \lambda g(\boldsymbol{x}^2).$$

**Proposition 2.4** (See, e.g., Luc [5], Proposition 6.2) *Let $X \subset \mathbb{R}^m$ be a nonempty convex subset. Then, a vector-valued function $g : X \to \mathbb{R}^n$ is $C$-convex on $X$ if and only if $\boldsymbol{c}^\top g(\cdot)$ is a real-valued convex function on $X$ for each $\boldsymbol{c} \in C^+$.*

**Definition 2.5** (**$C$-dc function**, see Hojo, Tanaka and Yamada [4]) Let $X \subset \mathbb{R}^m$ be nonempty and convex. A vector-valued function $f : X \to \mathbb{R}^n$ is said to be *$C$-dc* on $X$ if there exist two $C$-convex functions $g$ and $h$ on $X$ such that

$$f(\boldsymbol{x}) = g(\boldsymbol{x}) - h(\boldsymbol{x}) \tag{2}$$

for all $\boldsymbol{x} \in X$. Moreover, formulation (2) is called $C$-dc decomposition of $f$ over $X$.

**Definition 2.6** (**Efficiency**, see Luc [5], Definition 2.1) Let $X \subset \mathbb{R}^m$ be nonempty and let $f : X \to \mathbb{R}^n$ be a vector-valued function. We say that

(i) $\boldsymbol{y} \in X$ is an ideal efficient point of $f$ over $X$ with respect to $C$ if $f(\boldsymbol{y}) \leq_C f(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$.

(ii) $\boldsymbol{y} \in X$ is an efficient point of $f$ over $X$ with respect to $C$ if $f(\boldsymbol{x}) \leq_C f(\boldsymbol{y})$ for some $\boldsymbol{x} \in X$, then $f(\boldsymbol{y}) \leq_C f(\boldsymbol{x})$.

(iii) $\boldsymbol{y} \in X$ is a weakly efficient point of $f$ over $X$ with respect to $C$ if $f(\boldsymbol{x}) \leq_{\{0_n\}\cup \operatorname{int} C} f(\boldsymbol{y})$ for some $\boldsymbol{x} \in X$, then $f(\boldsymbol{y}) \leq_{\{0_n\}\cup \operatorname{int} C} f(\boldsymbol{x})$.

# 3 Optimality conditions for a cone-dc vector optimization problem

Let us consider the following mathematical programming problem.

$$(C\text{-DC}) \begin{cases} C\text{-min} & f(\boldsymbol{x}) := g(\boldsymbol{x}) - h(\boldsymbol{x}) \\ \quad \text{s.t.} & \boldsymbol{x} \in \mathbb{R}^m, \end{cases}$$

where $g, h : \mathbb{R}^m \to \mathbb{R}^n$ are $C$-dc functions and $C$-min denotes minimizing with respect to the ordering cone $C \subset \mathbb{R}^n$.

By applying Proposition 2.2 to ($C$-DC), we obtain the following theorems.

**Theorem 3.1** *A vector $\boldsymbol{y} \in \mathbb{R}^m$ is an ideal efficient point of ($C$-DC) if and only if*

$$\partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) \subset \partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top g(\boldsymbol{y}) \quad \text{for each } i \in \{1, \dots, l\} \text{ and } \varepsilon \in \mathbb{R}_+.$$

**Proof.** We assume that $\boldsymbol{y} \in \mathbb{R}^m$ is an ideal efficient point of ($C$-DC). From Definition 2.6, $f(\boldsymbol{x}) - f(\boldsymbol{y}) \in C$ for all $\boldsymbol{x} \in \mathbb{R}^m$. This implies that

$$\left(\boldsymbol{c}^i\right)^\top (f(\boldsymbol{x}) - f(\boldsymbol{y})) \geq 0 \quad \text{for each } \boldsymbol{x} \in \mathbb{R}^m \text{ and } i \in \{1, \dots, l\}.$$

Hence, we have

$$\left(\boldsymbol{c}^i\right)^\top f(\boldsymbol{y}) = \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y})) = \min\left\{\left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - h(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{R}^m\right\} \quad \text{for all } i \in \{1, \dots, l\}.$$

By Proposition 2.4, we note that $\left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{x})$ is convex for each $i \in \{1, \dots, l\}$. Hence, it follows from Proposition 2.2 that

$$\partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) \subset \partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top g(\boldsymbol{y}) \quad \text{for each } i \in \{1, \dots, l\} \text{ and } \varepsilon \in \mathbb{R}_+.$$

Next, we assume that

$$\partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) \subset \partial_\varepsilon \left(\boldsymbol{c}^i\right)^\top g(\boldsymbol{y}) \quad \text{for each } i \in \{1, \ldots, l\} \text{ and } \varepsilon \in \mathbb{R}_+.$$

From Proposition 2.2,

$$\left(\boldsymbol{c}^i\right)^\top f(\boldsymbol{y}) = \min\left\{\left(\boldsymbol{c}^i\right)^\top f(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^m\right\} \quad \text{for all } i \in \{1, \ldots, l\}.$$

Hence, we have

$$\left(\boldsymbol{c}^i\right)^\top (f(\boldsymbol{x}) - f(\boldsymbol{y})) \geq 0 \quad \text{for each } \boldsymbol{x} \in \mathbb{R}^m \text{ and } i \in \{1, \ldots, l\}.$$

From the definition of $C$, we have $f(\boldsymbol{x}) - f(\boldsymbol{y}) \in C$ for each $\boldsymbol{x} \in \mathbb{R}^m$. Therefore, $\boldsymbol{y} \in \mathbb{R}^m$ is an ideal efficient point of ($C$-DC). □

**Theorem 3.2** *A vector $\boldsymbol{y} \in \mathbb{R}^m$ is an efficient point of ($C$-DC) if and only if $\boldsymbol{y}$ satisfies*

$$\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{l} \bigcup_{\boldsymbol{a}^i \in \partial_{\varepsilon_i}(\boldsymbol{c}^i)^\top h(\boldsymbol{y})} \left\{\boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array}\right\}\right) \cap \left(\bigcup_{\boldsymbol{a}^j \in \partial_{\varepsilon_j}(\boldsymbol{c}^j)^\top h(\boldsymbol{y})} \left\{\boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_j \\ > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array}\right\}\right) = \emptyset$$

*for each $j \in \{1, \ldots, l\}$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top \in \mathbb{R}^n_+$.*

**Proof.** We assume that $\boldsymbol{y}$ is an efficient point of ($C$-DC). In order to obtain a contradiction, we suppose that there exists $\boldsymbol{x}' \in \mathbb{R}^m$ satisfying

$$\boldsymbol{x}' \in \left(\bigcap_{\substack{i=1 \\ i \neq j}}^{l} \bigcup_{\boldsymbol{a}^i \in \partial_{\varepsilon_i}(\boldsymbol{c}^i)^\top h(\boldsymbol{y})} \left\{\boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array}\right\}\right) \cap \left(\bigcup_{\boldsymbol{a}^j \in \partial_{\varepsilon_j}(\boldsymbol{c}^j)^\top h(\boldsymbol{y})} \left\{\boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_j \\ > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array}\right\}\right)$$

for some $j \in \{1, \ldots, l\}$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top \in \mathbb{R}^n_+$. Then, for all $i \in \{1, \ldots, l\} \backslash \{j\}$, there exists $\boldsymbol{a}^i \in \partial_{\varepsilon_i} \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y})$ such that

$$\left(\boldsymbol{c}^i\right)^\top (h(\boldsymbol{x}') - h(\boldsymbol{y})) \geq \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x}' - \boldsymbol{y}) - \varepsilon_i \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y})).$$

Moreover, there exists $\boldsymbol{a}^j \in \partial_{\varepsilon_j} \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y})$ such that

$$\left(\boldsymbol{c}^j\right)^\top (h(\boldsymbol{x}') - h(\boldsymbol{y})) \geq \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x}' - \boldsymbol{y}) - \varepsilon_j > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y})).$$

Hence, for all $i \in \{1, \ldots, l\}$,

$$\begin{array}{l} \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y})) \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - h(\boldsymbol{x}')) \text{ for all } i \in \{1, \ldots, l\} \backslash \{j\}, \\ \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y})) > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}') - h(\boldsymbol{x}')). \end{array}$$

This implies that

$$f(\boldsymbol{y}) - f(\boldsymbol{x}') \in C \text{ and } f(\boldsymbol{x}') - f(\boldsymbol{y}) \notin C.$$

This contradicts the efficiency of $\boldsymbol{y}$. Therefore,

$$\left( \bigcap_{\substack{i=1 \\ i \neq j}}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i} (c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right) \cap \left( \bigcup_{a^j \in \partial_{\varepsilon_j} (c^j)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_j \\ > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right) = \emptyset$$

for each $j \in \{1, \ldots, l\}$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top \in \mathbb{R}^n_+$.

Next, we assume that $\boldsymbol{y}$ satisfies

$$\left( \bigcap_{\substack{i=1 \\ i \neq j}}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i} (c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right) \cap \left( \bigcup_{a^j \in \partial_{\varepsilon_j} (c^j)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_j \\ > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right) = \emptyset$$

for each $j \in \{1, \ldots, l\}$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top \in \mathbb{R}^n_+$. In order to obtain a contradiction, we suppose that $\boldsymbol{y}$ is not efficient for ($C$-DC). Then, from Definition 2.6, there exists $\boldsymbol{x}' \in \mathbb{R}^m$ such that

$$f(\boldsymbol{y}) - f(\boldsymbol{x}') \in C \text{ and } f(\boldsymbol{x}') - f(\boldsymbol{y}) \notin C,$$

that is, $f(\boldsymbol{y}) - f(\boldsymbol{x}') \in C \backslash \{\mathbf{0}_n\}$. Hence, for each $i \in \{1, \ldots, l\}$,

$$\left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y})) \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - h(\boldsymbol{x}')).$$

Moreover, there exists $j \in \{1, \ldots, l\}$ such that

$$\left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y})) > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}') - h(\boldsymbol{x}')).$$

Let $\boldsymbol{a}^i \in \partial \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{x}')$ for each $i \in \{1, \ldots, l\}$. Since $\left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{x}')$ is a convex function for each $i \in \{1, \ldots, l\} \backslash \{j\}$, we have

$$0 \leq \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) - \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{x}') - \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{y} - \boldsymbol{x}') =: \varepsilon_i.$$

Moreover, we have

$$\begin{aligned} 0 &\leq \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) - \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{x}') - \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{y} - \boldsymbol{x}') \\ &< \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) - \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{x}') - \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{y} - \boldsymbol{x}') \\ &\quad + \frac{\left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} =: \varepsilon_j. \end{aligned}$$

Then, for each $\boldsymbol{z} \in \mathbb{R}^m$ and $i \in \{1, \dots, l\} \setminus \{j\}$,

$$\begin{aligned}
\left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{z}) &\geq \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{z} - \boldsymbol{x}') + \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{x}') \\
&= \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{z} - \boldsymbol{y}) + \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{y} - \boldsymbol{x}') + \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) \\
&\quad - \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) + \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{x}') \\
&= \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{z} - \boldsymbol{y}) + \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) - \varepsilon_i.
\end{aligned}$$

Moreover, for each $\boldsymbol{z} \in \mathbb{R}^m$,

$$\begin{aligned}
\left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{z}) &\geq \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{z} - \boldsymbol{x}') + \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{x}') \\
&= \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{z} - \boldsymbol{y}) + \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{y} - \boldsymbol{x}') + \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) \\
&\quad - \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) + \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{x}') \\
&= \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{z} - \boldsymbol{y}) + \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{y} - \boldsymbol{x}') + \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) \\
&\quad - \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) + \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{x}') \\
&\quad - \frac{\left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} \\
&= \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{z} - \boldsymbol{y}) + \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) - \varepsilon_j.
\end{aligned}$$

Hence, $\boldsymbol{a}^i \in \partial_{\varepsilon_i} \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y})$ for each $i \in \{1, \dots, l\}$. Here, we have

$$\begin{aligned}
&\left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x}' - \boldsymbol{y}) - \varepsilon_i - \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y}))) \\
&= \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x}' - \boldsymbol{y}) - \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y}))) - \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y}) + \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{x}') \\
&\quad + \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{y} - \boldsymbol{x}') \\
&= \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}')) \geq 0
\end{aligned}$$

for each $i \in \{1, \dots, l\} \setminus \{j\}$. Moreover,

$$\begin{aligned}
&\left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x}' - \boldsymbol{y}) - \varepsilon_j - \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y}))) \\
&= \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x}' - \boldsymbol{y}) - \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y}))) - \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{y}) + \left(\boldsymbol{c}^j\right)^\top h(\boldsymbol{x}') \\
&\quad + \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{y} - \boldsymbol{x}') - \frac{\left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} \\
&= \frac{\left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} > 0
\end{aligned}$$

for each $i \in \{1, \dots, l\}$. This implies that

$$\begin{aligned}
\boldsymbol{x}' \in &\left( \bigcap_{\substack{i=1 \\ i \neq j}}^{l} \bigcup_{\boldsymbol{a}^i \in \partial_{\varepsilon_i} (\boldsymbol{c}^i)^\top h(\boldsymbol{y})} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ \geq \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right) \\
&\bigcap \left( \bigcup_{\boldsymbol{a}^j \in \partial_{\varepsilon_j} (\boldsymbol{c}^j)^\top h(\boldsymbol{y})} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^j\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_j \\ > \left(\boldsymbol{c}^j\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} \right).
\end{aligned}$$

This is a contradiction. Therefore, $\boldsymbol{y}$ is an efficient point of ($C$-DC). $\square$

**Theorem 3.3** *A vector $\boldsymbol{y} \in \mathbb{R}^m$ is a weakly efficient point of (C-DC) if and only if $\boldsymbol{y}$ satisfies*

$$\bigcap_{i=1}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i}(c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} = \emptyset$$

*for each $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)^\top \in \mathbb{R}^n_+$.*

**Proof.** We assume that $\boldsymbol{y} \in \mathbb{R}^m$ is a weakly efficient point of ($C$-DC). In order to obtain a contradiction, we suppose that there exists $\boldsymbol{x}' \in \mathbb{R}^m$ satisfying

$$\boldsymbol{x}' \in \bigcap_{i=1}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i}(c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\}$$

for some $\boldsymbol{\varepsilon} \in \mathbb{R}^n_+$. Then, for all $i \in \{1, \dots, l\}$, there exists $\boldsymbol{a}^i \in \partial_{\varepsilon_i} \left(\boldsymbol{c}^i\right)^\top h(\boldsymbol{y})$ such that

$$\left(\boldsymbol{c}^i\right)^\top (h(\boldsymbol{x}') - h(\boldsymbol{y})) \geq \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x}' - \boldsymbol{y}) - \varepsilon_i > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y})).$$

Hence, for all $i \in \{1, \dots, l\}$,

$$\left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y})) > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - h(\boldsymbol{x}')).$$

This implies that

$$f(\boldsymbol{y}) - f(\boldsymbol{x}') \in \{\mathbf{0}_n\} \cup \operatorname{int} C \text{ and } f(\boldsymbol{x}') - f(\boldsymbol{y}) \notin \{\mathbf{0}_n\} \cup \operatorname{int} C.$$

This contradicts the weak efficiency of $\boldsymbol{y}$. Therefore,

$$\bigcap_{i=1}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i}(c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} = \emptyset$$

for each $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)^\top \in \mathbb{R}^n_+$.

Next, we assume $\boldsymbol{y} \in \mathbb{R}^m$ satisfying

$$\bigcap_{i=1}^{l} \bigcup_{a^i \in \partial_{\varepsilon_i}(c^i)^\top h(y)} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} \left(\boldsymbol{a}^i\right)^\top (\boldsymbol{x} - \boldsymbol{y}) - \varepsilon_i \\ > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\} = \emptyset$$

for each $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)^\top \in \mathbb{R}^n_+$. In order to obtain a contradiction, we suppose that $\boldsymbol{y}$ is not a weakly efficient point of ($C$-DC). Then, from Definition 2.6, there exists $\boldsymbol{x}' \in \mathbb{R}^m$ such that

$$f(\boldsymbol{y}) - f(\boldsymbol{x}') \in \{\mathbf{0}_n\} \cup \operatorname{int} C \text{ and } f(\boldsymbol{x}') - f(\boldsymbol{y}) \notin \{\mathbf{0}_n\} \cup \operatorname{int} C,$$

that is, $f(\boldsymbol{y}) - f(\boldsymbol{x}') \in \operatorname{int} C$. Hence, for each $i \in \{1, \dots, l\}$,

$$\left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y})) > \left(\boldsymbol{c}^i\right)^\top (g(\boldsymbol{x}') - h(\boldsymbol{x}')).$$

Let $\boldsymbol{a}^i \in \partial (\boldsymbol{c}^i)^\top h(\boldsymbol{x}')$ for each $i \in \{1,\dots,l\}$. Since $(\boldsymbol{c}^i)^\top h(\boldsymbol{x}')$ is a convex function for each $i \in \{1,\dots,l\}$, we have

$$\begin{aligned}0 &\leq (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) - (\boldsymbol{c}^i)^\top h(\boldsymbol{x}') - (\boldsymbol{a}^i)^\top (\boldsymbol{y}-\boldsymbol{x}') \\ &< (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) - (\boldsymbol{c}^i)^\top h(\boldsymbol{x}') - (\boldsymbol{a}^i)^\top (\boldsymbol{y}-\boldsymbol{x}') \\ &\quad + \frac{(\boldsymbol{c}^i)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} =: \varepsilon_i.\end{aligned}$$

Then, for each $\boldsymbol{z} \in \mathbb{R}^m$ and $i \in \{1,\dots,l\}$,

$$\begin{aligned}(\boldsymbol{c}^i)^\top h(\boldsymbol{z}) &\geq (\boldsymbol{a}^i)^\top (\boldsymbol{z}-\boldsymbol{x}') + (\boldsymbol{c}^i)^\top h(\boldsymbol{x}') \\ &= (\boldsymbol{a}^i)^\top (\boldsymbol{z}-\boldsymbol{y}) + (\boldsymbol{a}^i)^\top (\boldsymbol{y}-\boldsymbol{x}') + (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) \\ &\quad - (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) + (\boldsymbol{c}^i)^\top h(\boldsymbol{x}') \\ &= (\boldsymbol{a}^i)^\top (\boldsymbol{z}-\boldsymbol{y}) + (\boldsymbol{a}^i)^\top (\boldsymbol{y}-\boldsymbol{x}') + (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) \\ &\quad - (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) + (\boldsymbol{c}^i)^\top h(\boldsymbol{x}') \\ &\quad - \frac{(\boldsymbol{c}^i)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} \\ &= (\boldsymbol{a}^i)^\top (\boldsymbol{z}-\boldsymbol{y}) + (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) - \varepsilon_i.\end{aligned}$$

Hence, $\boldsymbol{a}^i \in \partial_{\varepsilon_i} (\boldsymbol{c}^i)^\top h(\boldsymbol{y})$ for each $i \in \{1,\dots,l\}$. Moreover, we have

$$\begin{aligned}&(\boldsymbol{a}^i)^\top (\boldsymbol{x}'-\boldsymbol{y}) - \varepsilon_i - (\boldsymbol{c}^i)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y}))) \\ &= (\boldsymbol{a}^i)^\top (\boldsymbol{x}'-\boldsymbol{y}) - (\boldsymbol{c}^i)^\top (g(\boldsymbol{x}') - g(\boldsymbol{y}))) - (\boldsymbol{c}^i)^\top h(\boldsymbol{y}) + (\boldsymbol{c}^i)^\top h(\boldsymbol{x}') \\ &\quad + (\boldsymbol{a}^i)^\top (\boldsymbol{y}-\boldsymbol{x}') - \frac{(\boldsymbol{c}^i)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} \\ &= \frac{(\boldsymbol{c}^i)^\top (g(\boldsymbol{y}) - h(\boldsymbol{y}) - g(\boldsymbol{x}') + h(\boldsymbol{x}'))}{2} > 0\end{aligned}$$

for each $i \in \{1,\dots,l\}$. This implies that

$$\boldsymbol{x}' \in \bigcap_{i=1}^{l} \bigcup_{\boldsymbol{a}^i \in \partial_{\varepsilon_i} (\boldsymbol{c}^i)^\top h(\boldsymbol{y})} \left\{ \boldsymbol{x} \in \mathbb{R}^m : \begin{array}{l} (\boldsymbol{a}^i)^\top (\boldsymbol{x}-\boldsymbol{y}) - \varepsilon_i \\ > (\boldsymbol{c}^i)^\top (g(\boldsymbol{x}) - g(\boldsymbol{y})) \end{array} \right\}.$$

This is a contradiction. Therefore, $\boldsymbol{y}$ is a weakly efficient point of ($C$-DC). □

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